FORELESNING 31 — METI180 MATEMATIKK

BI

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Plan:

Gjennomgå prøve-eksamen 05/2016

Repetisjon, Om eksamen

Om eksamen..

* Les spørsmålet. Svar på spørsmålet.
* Alle svar skal begrunnes. Korte og presise begrunnelser er best.
* Disponer tida. Svar på lette oppgaver først.
* Sjekk formelsamling.

Karakterskala:

$16 \times 6p = 96p = 100\%$ (uten bonus)

Anbefalt karakterskala: (dvs utgangspunkt, men ikke sikkert karakterskalaen blir helt slik)

A	92%
B	77%
C	58%
D	46%
E	40%

1. $A = \begin{pmatrix} 2-r & 2 & -1 \\ 1 & 3-r & -1 \\ -1 & -2 & 2-r \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 3 \\ 3 \\ r-1 \end{pmatrix}$

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a) $\underline{r=5}$:

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & 3 \\ 1 & -2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -3 & 2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right) \begin{array}{c} 3 \\ 1 \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & -4 & -4 & 7 \end{array}\right) \begin{array}{c} \\ -1 \\ \end{array} \to \left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

ingen løsning.
(pga pivot-posisjon i siste kolonne)

Frihetsgrader:
ingen.

b) $\begin{vmatrix} 2-r & 2 & -1 \\ 1 & 3-r & -1 \\ -1 & -2 & 2-r \end{vmatrix}$

$= (2-r) \cdot \begin{vmatrix} 3-r & -1 \\ -2 & 2-r \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & -1 \\ -2 & 2-r \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & -1 \\ 3-r & -1 \end{vmatrix}$

$= (2-r) \cdot \left((3-r)(2-r) - 2 \right) - \left(2(2-r) - 2 \right) - \left(-2 + 3 - r \right)$

$= (2-r) \cdot \left((3-r)(2-r) - 2 \right) + \underbrace{3r - 3}_{= 3(r-1)}$

$= (2-r) \cdot \left(r^2 - 5r + 4 \right) + 3(r-1)$

$= (2-r)(r-1)(r-4) + 3(r-1)$

$r^2 - 5r + 4 = 0$
$r = 1, r = 4$

$$= (r-1)\cdot\left[(2-r)(r-4) + 3\right]$$

$$= (r-1)\cdot(-r^2+6r-5)$$

$$= (r-1)\cdot(r-1)(r-5)\cdot(-1)$$

$$= \underline{\underline{-(r-1)^2(r-5)}}$$

$-r^2+6r-5=0$
$r=1,\ r=5$

Alt.: Uten faktorisering

$$|A| = (2-r)\cdot\left((3-r)(2-r)-2\right) + 3r-3$$

$$= (2-r)(r^2-5r+4) + 3r-3$$

$$= -r^3+7r^2-14r+8+3r-3$$

$$= \underline{\underline{-r^3+7r^2-11r+5}}$$

c) $\underline{r=0}$:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix} \qquad |A| = 5$$

$$A^{-1} = \frac{1}{|A|}\cdot \operatorname{adj}(A) = \frac{1}{|A|}\cdot\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ \vdots & \vdots & \vdots \end{pmatrix}^T$$

$C_{11} = 4 \quad C_{12} = -1 \quad C_{13} = 1$

$C_{21} = -2 \quad C_{22} = 3 \quad C_{23} = 2 \qquad \Rightarrow A^{-1} = \frac{1}{5}\cdot\begin{pmatrix} 4 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$

$C_{31} = 1 \quad C_{32} = 1 \quad C_{33} = 4$

Matriselikn:

$$A\underline{x} = \underline{b} \Rightarrow \underline{x} = A^{-1}\cdot\underline{b} = \frac{1}{5}\begin{pmatrix} 4 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}}$$

d) $|A| \neq 0 \iff$ systemet har én løsning

<u>$|A| = 0$:</u> $\quad -(r-1)^2(r-5) = 0 \iff \underline{r=1}, \underline{r=5}$

<u>Konklusjon:</u> én løsn. når $r \neq 1, 5$

<u>Alt:</u> $\quad -r^3 + 7r^2 - 11r + 5 = 0$

- Hvis likn. har heltallig løsn., så må den være faktor i 5, dvs: $\pm 1, \pm 5$. Når man har funnet én løsn, kan man bruke polynomdiv. til å finne de andre.
- Faktisk er $r=5$ én løsn. pga. a).

<u>Finn y når $r \neq 1, 5$:</u> Cramers regel

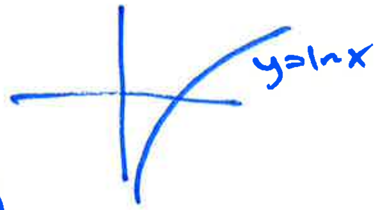
$$y = \frac{|A_2(\underline{b})|}{|A|} \qquad A = \begin{pmatrix} 2-r & 2 & -1 \\ 1 & 3-r & -1 \\ -1 & -2 & 2-r \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ 3 \\ r-1 \end{pmatrix}$$

$$|A_2(\underline{b})| = \begin{vmatrix} 2-r & 3 & -1 \\ 1 & 3 & -1 \\ -1 & r-1 & 2-r \end{vmatrix} = (2-r)\cdot(3(2-r)+r-1)$$
$$-1\cdot(3(2-r)+r-1)$$
$$\cancel{-1\cdot(-3+3)}$$

$$= (2-r)\cdot(5-2r) - 1\cdot(5-2r) = \underline{(1-r)(5-2r)}$$

$$y = \frac{\cancel{(1-r)}(5-2r)}{\cancel{-}(r-1)^{\cancel{2}}(r-5)} = \underline{\underline{\frac{5-2r}{(r-1)(r-5)}}}, \quad r \neq 1,5$$

2. $f(x) = x \ln x \quad , \quad x > 0$



$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x \cdot \ln x = \underline{\underline{\infty}}$$

(fordi $x \to \infty \Rightarrow \ln x \to \infty$)

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \underset{\uparrow}{x \cdot \ln x} = \lim_{x \to 0^+} \underset{\uparrow}{\frac{\ln x}{1/x}}$$

"$0 \cdot \infty$" ubestemt $\quad$ "$\frac{\infty}{\infty}$"

L'Hop.

$$= \lim_{x \to 0^+} \frac{1/x \cdot x^2}{-1/x^2 \cdot x^2} = \lim_{x \to 0^+} \frac{x}{-1} = \underline{\underline{0}}$$

3.

a) $\displaystyle \int \frac{\ln x + 1}{x^2}\, dx = \int \overset{u'}{\frac{1}{x^2}} \cdot \overset{v}{(\ln x + 1)}\, dx$

Delvis: $\displaystyle \int u' v \, dx = uv - \int u v' \, dx$

$u = -1/x \qquad v = \ln x + 1$

$u' = 1/x^2 \qquad v' = 1/x$

$$= -\frac{1}{x} \cdot (\ln x + 1) - \int \left(-\frac{1}{x}\right) \cdot \frac{1}{x}\, dx$$

$$= -\frac{1}{x} \cdot (\ln x + 1) + \int \frac{1}{x^2}\, dx = \underline{-\frac{1}{x}(\ln x + 1) - \frac{1}{x} + C}$$

Husk: $\ln x + 1 = \ln(x) + 1$

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b) $$\int x^3\sqrt{x^2+4}\,dx = \int x^3\sqrt{u}\,\frac{du}{2x}$$

$$\boxed{\begin{array}{l}\underline{\text{Subst:}}\\ u = x^2+4\\ du = 2x\cdot dx\end{array}} \longrightarrow x^2 = u-4$$

$$= \int \frac{1}{2}x^2\sqrt{u}\,du = \int \frac{1}{2}(u-4)\sqrt{u}\,du$$

$$= \frac{1}{2}\int u^{3/2} - 4u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{5}\cdot u^{5/2} - 4\cdot\frac{2}{3}\cdot u^{3/2}\right) + C$$

$$= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

c) $$\int \frac{x^2}{x^2-5x+4}\,dx = \int 1 + \frac{5x-4}{x^2-5x+4}\,dx$$

$$\boxed{\begin{array}{l} x^2 : (x^2-5x+4) = \underline{1}\\ -(x^2-5x+4)\\ \hline 5x-4\end{array}}$$

$$\frac{5x-4}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} \quad | \cdot (x-4)(x-1)$$

$$5x-4 = A\cdot(x-1) + B(x-4)$$

$\underline{x=1}$: $1 = B\cdot(-3)$, $\underline{B = -1/3}$

$\underline{x=4}$: $16 = A\cdot 3$, $\underline{A = 16/3}$

$$= x + \int \frac{5x-4}{x^2-5x+4}\,dx$$

$$= x + \int \frac{16/3}{x-4} + \frac{-1/3}{x-1}\,dx$$

Husk: $\ln a + \ln b = \ln(ab)$

$\ln(a) - \ln b = \ln(a/b)$

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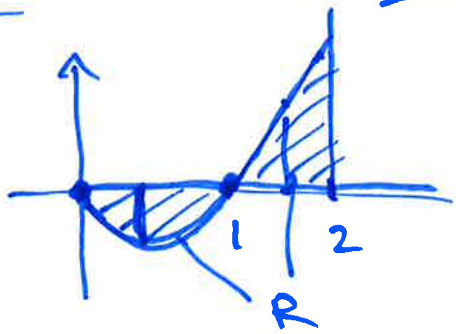
$$= x + \frac{16}{3}\ln|x-4| - \frac{1}{3}\cdot\ln|x-1| + C$$

d) $f(x) = x\ln x$ $\qquad$ $f(x)=0:\ x=0,\ x=1$



$$A = A_1 + A_2$$

$$= \int_0^1 -f(x)\,dx + \int_1^2 f(x)\,dx$$

$$\int x\ln x\,dx = \frac{1}{2}x^2\cdot\ln x - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\cdot\frac{1}{2}x^2 + C = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

$$A = -\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_0^1 + \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^2$$

$$= -\left(-\frac{1}{4} - 0\right) + \left(2\ln 2 - 1 - (-1/4)\right)$$

$$= 1/4 + 2\ln 2 - 1 + 1/4$$

$$= 2\ln 2 - 1/2$$

Egentlig: $\lim_{x\to 0^+} \frac{1}{2}x^2\ln x = 0$

4. $f(x,y) = (xy+2x)\,e^{x+y}$

a) $f'_x = (y+2)\cdot e^{x+y} + (xy+2x)\cdot e^{x+y}\cdot 1$

$= (\underline{y+2} + \underline{xy+2x})\,e^{x+y} = 0$

$f'_y = (\underline{x} + \underline{xy+2x})\,e^{x+y} = 0$

$y+2+xy+2x = 0$

$x + xy + 2x = 0 \longrightarrow 3x+xy=0$

$x\cdot(3+y)=0$

$\underline{x=0}$ eller $y=\underline{-3}$

$y+2=0$	$-1-x=0$
$y=-2$	$x=-1$
$\underline{(0,-2)}$	$\underline{(-1,-3)}$

Stasj. pkt:

$(x,y) = \underline{(0,-2)},\ \underline{(-1,-3)}$

$f'_x = (y+2+xy+2x)\,e^{x+y}$ $\quad f'_y = (3x+xy)\,e^{x+y}$

b) $f''_{xx} = (y+2+y+2+xy+2x)\,e^{x+y}$

$= \underline{(2y+4+xy+2x)\,e^{x+y}}$

$f''_{xy} = (1+x+y+2+xy+2x)\,e^{x+y}$

$= \underline{(3+3x+y+xy)\,e^{x+y}}$

$f''_{yy} = (x+3x+xy)\,e^{x+y} = \underline{(4x+xy)\,e^{x+y}}$

$(x,y) = (0,-2)$:

$$H(f)(0,-2) = \begin{pmatrix} 0 & 1\cdot e^{-2} \\ 1\cdot e^{-2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-2} \\ e^{-2} & 0 \end{pmatrix}$$

$\det = 0 - e^{-4} < 0 \Rightarrow$ $(0,-2)$ sadelpkt

$(x,y) = (-1,-3)$:

$$H(f)(-1,-3) = \begin{pmatrix} -1\cdot e^{-4} & 0 \\ 0 & -1\cdot e^{-4} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\det = e^{-8} - 0 = e^{-8} > 0$ $\leftarrow$ $AC-B^2$

$A = -1\cdot e^{-4} < 0$ $(-1,-3)$ lokalt maks

Husk: $AC-B^2 > 0,\ A > 0$: lok. min
$AC-B^2 > 0,\ A < 0$: lok. maks.
$AC-B^2 < 0$: sadelpkt.

$f = (xy+2x)e^{x+y}$

c) $L(x,y) = f(2,-2) + f'_x(2,-2)\cdot(x-2) + f'_y(2,-2)\cdot(y+2)$
$= 0 + 0\cdot(x-2) + 2\cdot(y+2)$
$= \underline{\underline{2y+4}}$

d) Globale maks/min:

Globalt min: Ingen lokale min fra b)
$\Rightarrow$ ingen globale min.

Globale maks: $(-1,-3)$ lokalt maks $f(-1,-3) = 1\cdot e^{-4} = 1/e^4$

$f(x,y) = (xy+2x)e^{x+y}$ $f(1,1) = 3e^2 > 1/e^4$
$\Rightarrow$ Ingen globale maks.

5. maks/min $f(x,y) = \ln(16 - x^2 - y^2)$ når $2x + 3y = 12$

a) $D_f = \{(x,y) : 16 - x^2 - y^2 > 0\} = \{(x,y) : x^2 + y^2 < 16\}$

$x^2 + y^2 < 16$ må være oppfylt for at f skal være defn.

Skisse: Se nedenfor

b) $L = \ln(16 - x^2 - y^2) - \lambda \cdot (2x + 3y)$ $\qquad u = 16 - x^2 - y^2$

FOC
$$\begin{cases} L'_x = \dfrac{-2x}{16 - x^2 - y^2} - \lambda \cdot 2 = 0 \\ L'_y = \dfrac{-2y}{16 - x^2 - y^2} - \lambda \cdot 3 = 0 \end{cases}$$

C $\qquad 2x + 3y = 12$

$\dfrac{-2x}{u} = 2\lambda$

$\dfrac{-2y}{u} = 3\lambda$

$2x + 3y = 12$

$$\left.\begin{aligned} -2x &= 2\lambda u \\ -2y &= 3\lambda u \end{aligned}\right\} \quad \lambda u = -\frac{2x}{2} = -\frac{2y}{3} \quad | \cdot 6$$

$-6x = -4y$

$2x + 3y = 12$ $\qquad y = \dfrac{-6x}{-4} = \underline{\dfrac{3}{2}x}$

$2x + 3 \cdot \frac{3}{2}x = 12 \quad | \cdot 2$

$4x + 9x = 24$

$13x = 24 \Rightarrow x = \underline{24/13}$ $\qquad y = \underline{36/13}$

Kandidatpkt uten Lagranges metode: $(x,y) = \underline{\underline{(24/13, 36/13)}}$

$$\left(\lambda = -\frac{2x}{2u} = -\frac{24/13}{16 - (24/13)^2 - (36/13)^2} \cdot \frac{13^2}{13^2} \right)$$

$$= -\frac{24 \cdot 13}{16 \cdot 13^2 - 24^2 - 36^2} = -\frac{312}{832} = -\frac{39}{104} = \underline{-0{,}375}$$

c) Husk:

Alternativer for Lagrange-problem:

i) Lagrange's metode (som i b))

ii) Løs bibet. for en var. og sett inn i funksjonen (som i c))

$\boxed{\text{iii) Som i een:} \quad \begin{aligned} &\min\ x^2+4xy+5y^2 \text{ når } xy=1 \\ &= \min\ x^2+4\cdot 1+5y^2 \text{ når } xy=1 \\ &= \min\ x^2+4+5y^2 \text{ når } xy=1 \end{aligned}}$

ii) max/min $f=\ln(16-x^2-y^2)$ når $2x+3y=12$

$y=\frac{12-2x}{3}$

$\underline{y=\frac{2}{3}(6-x)}$

$f=\ln\left(16-x^2-\frac{4}{9}(6-x)^2\right)$

$=\ln\left(\cancel{16}-x^2-\cancel{\tfrac{4}{9}}(36-12x+x^2)\right)$

$f=\ln\left(\frac{48}{9}x-\frac{13}{9}x^2\right)$

max/min $f(x)=\ln\left(\frac{48}{9}x-\frac{13}{9}x^2\right)$

$f'=\frac{1}{\frac{48}{9}x-\frac{13}{9}x^2}\cdot\left(\frac{48}{9}-\frac{26}{9}x\right) \quad \cdot\frac{9}{9}$

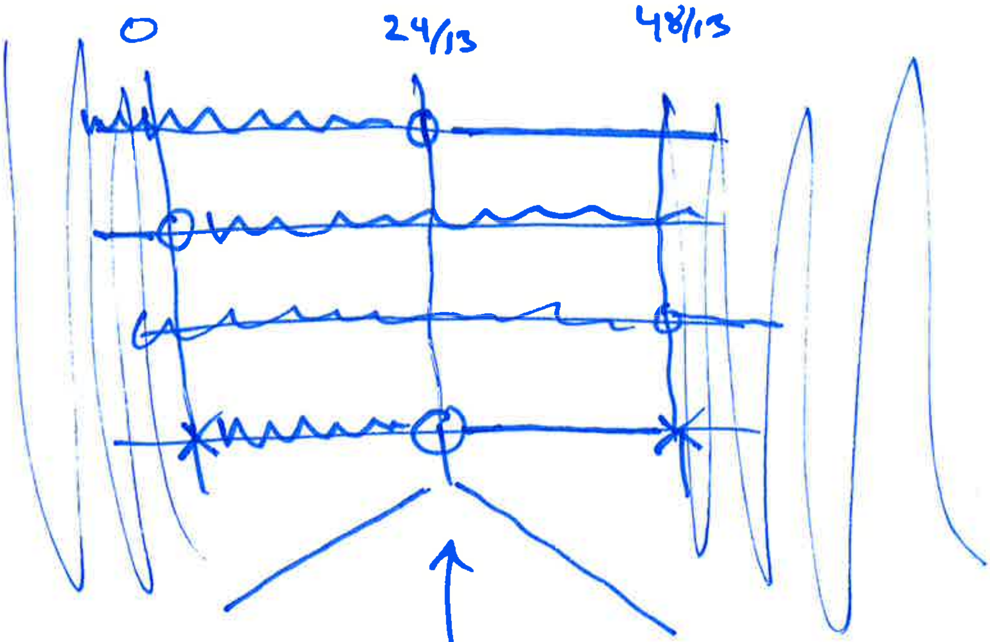
$f'=\frac{48-26x}{48x-13x^2}=0$

$48-26x=0$

$26x=48$

$x=48/26=\underline{24/13}$

$$f' = \frac{48-26x}{48x-13x^2} = \frac{48-26x}{x(48-13x)}$$



$x = 24/13$

maks.

$$16 - x^2 - y^2 > 0$$

$$x^2 + y^2 < 16$$

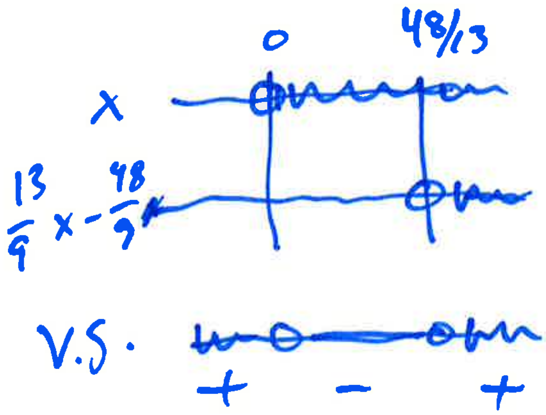
$$x^2 + \frac{4}{9}(36 - 12x + x^2) < 16$$

$$\frac{13}{9}x^2 - \frac{48}{9}x < 0$$

Løsn. av ulikhet

$(0, 48/13)$

$$x\left(\frac{13}{9}x - \frac{48}{9}\right) < 0$$



Løsn: $(0, 48/13)$

Konklusjon:

Maks: $x = 24/13$

$f(24/13) = \ln\left(\frac{48}{9}\cdot\frac{24}{13} - \frac{13}{9}\cdot\left(\frac{24}{13}\right)^2\right)$

$= \ln\left(\frac{128}{13} - \frac{64}{13}\right) = \ln\left(\frac{64}{13}\right)$

Min: Ikke defn. i $x=0$, $x = 48/13$

$f \to -\infty$ når $x \to 0^+$, $x \to 48/13^-$

$\Downarrow$

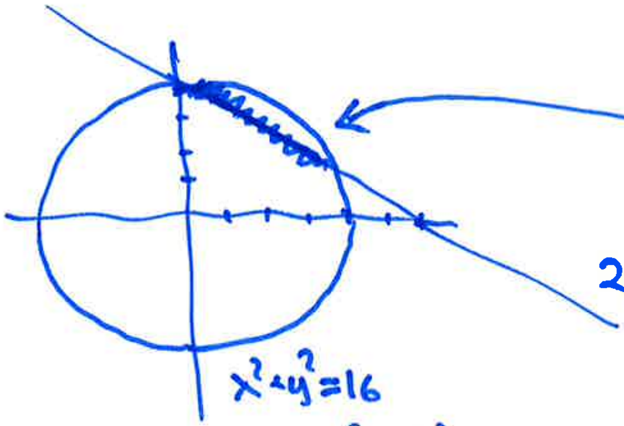
ikke noe minimum

Skisse for a):

Tillatte pkt:

$2x+3y=12$ (bibetingelse)

$x^2+y^2<16$ (D_f)



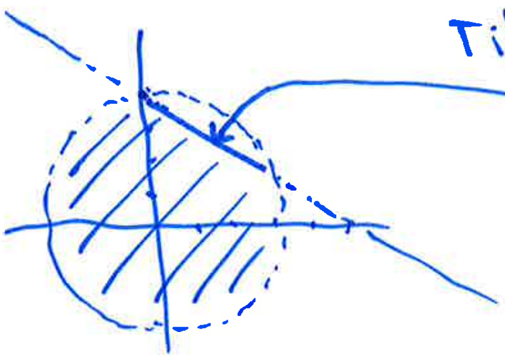
Tillatte pkt:

den delen av linjen som er innenfor sirkelen:

$2x+3y=12$

$x^2+y^2<16$

d) Bonus: $2x+3y \leq 12$



Tillatte pkt:

I

$\boxed{\begin{array}{l} 2x+3y=12 \\ x^2+y^2<16 \end{array}}$

som i a)-c)

II

$\boxed{\begin{array}{l} 2x+3y<12 \\ x^2+y^2<16 \end{array}}$

innenfor sirkelen, under linjen

Kandidater på linja: I

Som for a) - c): $\boxed{\begin{array}{c}(x,y)=(24/13,\ 36/13)\\ f=\ln(64/13)\end{array}}$ kandidat for maks.

Kandidater under linjen: II

Må finne disse, det er stasjonære pkt. for f med $2x+3y<12$, $x^2+y^2<16$:

$$f=\ln(16-x^2-y^2)$$

$$\left.\begin{array}{ll} f'_x=\dfrac{1}{16-x^2-y^2}\cdot(-2x)=0 & \Rightarrow x=0\\ f'_y=\dfrac{1}{16-x^2-y^2}\cdot(-2y)=0 & \Rightarrow y=0 \end{array}\right\}$$

Stasj. pkt: $(x,y)=(0,0)$

$2x+3y=0<12$
$x^2+y^2=0<16$

ok (under linjen, innenfor sirkelen)

$f(0,0)=\ln(16)$

Konkl:

Siden $\underset{\simeq 2{,}77}{f(0,0)=\ln(16)} > \underset{\simeq 1{,}59}{f(24/13,36/13)=\ln(64/13)}$,

er $\boxed{\begin{array}{c}(x,y)=(0,0)\\ f(0,0)=\ln(16)\end{array}}$ maks. (det fins maks pga ekstremverdisetn + d))

Siden $f\to-\infty$ når $(x,y)\to$ sirkelen, er det ikke noe minimum.