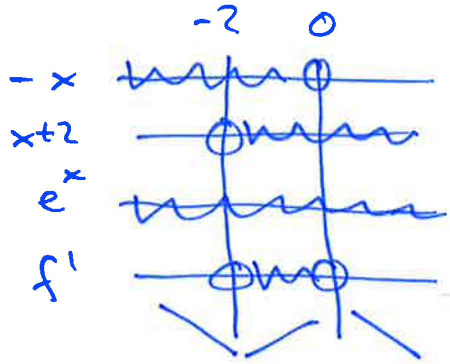
Løsning Innlevering III.

1. $f(x) = 1 - x^2 e^x$

a) $f' = -2xe^x - x^2e^x = \underline{-x(x+2)e^x}$

$f'' = -2e^x - 2xe^x - 2xe^x - x^2e^x = \underline{-(x^2+4x+2)e^x}$

b) Fortegnsskjema for f':



Min: $\lim_{x\to\infty} f(x) = -\infty$

ikke noe globalt min.

Max: $f(0) = 1$

$$\lim_{x\to-\infty} f(x) = 1 - \lim_{x\to-\infty} \frac{x^2}{e^{-x}} = 1 - \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

$$= 1 - \lim_{x\to\infty} \frac{2}{e^{-x}} = \underline{1}$$

Globalt max: $\underline{\underline{f=1}}$ i $\underline{x=0}$

c) $f'' = -(x^2+4x+2)e^x = 0$

$x^2+4x+2=0$

$$x = \frac{-4 \pm \sqrt{16-4\cdot 2}}{2}$$

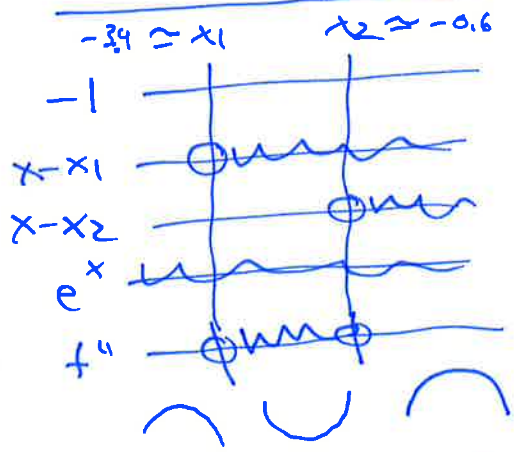
$$= \frac{-4\pm\sqrt{8}}{2} = -2 \pm \sqrt{2}$$

$2\sqrt{2}$

$x_1 = -2-\sqrt{2} \simeq -3.4$

$x_2 = -2+\sqrt{2} \simeq -0.6$

Fortegnsskjema for f'':



Vendepkt: $x_1 = \underline{\underline{-2-\sqrt{2}}} \simeq -3.4$

og

$x_2 = \underline{\underline{-2+\sqrt{2}}} \simeq -0.6$

d) f hverken monotont voksende / avtagende

$\Downarrow$

f har <u>ikke</u> omvendt funksjon

2. a) $\displaystyle\int (1-x)^4\,dx = \int u^4 \frac{du}{-1} = -\frac{1}{5}u^5 + C = \underline{-\frac{1}{5}(1-x)^5 + C}$

$\boxed{\begin{array}{l} u = 1-x \\ du = -dx \end{array}}$

b) $\displaystyle\int \frac{1-x}{x^3}\,dx = \int \frac{1}{x^3} - \frac{1}{x^2}\,dx = \int x^{-3} - x^{-2}\,dx = -\frac{1}{2}x^{-2} + x^{-1} + C$

$\displaystyle = \underline{-\frac{1}{2x^2} + \frac{1}{x} + C}$

c) $\displaystyle\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \underline{\frac{2}{5}x^{5/2} + C}$

d) $\displaystyle\int \frac{e^x + e^{-x}}{e^x}\,dx = \int 1 + e^{-2x}\,dx = \underline{x - \frac{1}{2}e^{-2x} + C}$

3. a) $\displaystyle\int \frac{3}{x^2 - 7x + 12}\,dx = \int \frac{3}{x-4} - \frac{3}{x-3}\,dx = 3\ln|x-4| - 3\ln|x-3| + C$

$\displaystyle = \underline{3\ln\left|\frac{x-4}{x-3}\right| + C}$

$x^2 - 7x + 12 = 0$

$x = 3, 4$

$\Downarrow$

$\displaystyle \frac{3}{x^2-7x+12} = \frac{3}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4} = \underline{\frac{-3}{x-3} + \frac{3}{x-4}}$

$3 = A(x-4) + B(x-3)$

$= \underbrace{(A+B)}_{=0}x + \underbrace{(-4A-3B)}_{=3}$

$B = -A$ $\quad -4A + 3A = 3$

$-A = 3$

$A = -3$

$B = 3$

b) $\int x^2 e^{-x}\,dx = -x^2e^{-x} - \int -2xe^{-x}\,dx$

$$\boxed{\begin{matrix} u=-e^{-x} & v=x^2 \\ u'=e^{-x} & v'=2x \end{matrix}}$$

$$= -x^2e^{-x} + \int 2xe^{-x}\,dx = -x^2e^{-x} + \left(-2xe^{-x} - \int -2e^{-x}\,dx\right)$$

$$\boxed{\begin{matrix} u=-e^{-x} & v=2x \\ u'=e^{-x} & v'=2 \end{matrix}}$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx = \underline{-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C}$$

c) $\int \frac{e^x}{e^x+e^{-x}}\,dx = \int \frac{\cancel{e^x}}{u+1/u}\cdot\frac{du}{\cancel{e^x}} = \int \frac{1\cdot u}{(u+1/u)\cdot u}\,du$

$$\boxed{\begin{matrix} u=e^x \\ du=e^x\,dx \end{matrix}}$$

$$= \int \frac{u}{u^2+1}\,du =$$

$$\boxed{\begin{matrix} v=u^2+1 \\ dv=2u\,du \end{matrix}}$$

$$= \int \frac{\cancel{u}}{v}\cdot\frac{dv}{2\cancel{u}} = \frac{1}{2}\int \frac{1}{v}\,dv$$

$$= \frac{1}{2}\ln|v| + C = \frac{1}{2}\ln(u^2+1) + C$$

$$= \underline{\underline{\frac{1}{2}\ln\left(e^{2x}+1\right) + C}}$$

d) $\int x\sqrt{x}\ln x\,dx$

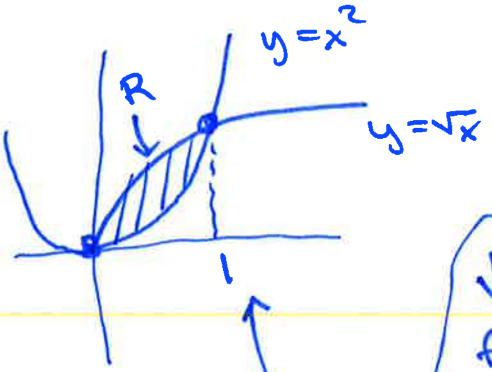
$$= \int x^{3/2}\cdot\ln x\,dx = \frac{2}{5}x^{5/2}\cdot\ln x - \int \frac{2}{5}x^{5/2}\cdot\frac{1}{x}\,dx$$

$$\boxed{\begin{matrix} u=\frac{2}{5}x^{5/2} & v=\ln x \\ u'=x^{3/2} & v'=1/x \end{matrix}}$$

$$= \frac{2}{5}x^2\sqrt{x}\ln x - \frac{2}{5}\int x^{3/2}\,dx = \frac{2}{5}x^2\sqrt{x}\ln x - \frac{2}{5}\cdot\left(\frac{2}{5}x^{5/2}\right) + C$$

$$= \underline{\frac{2}{5}x^2\sqrt{x}\ln x - \frac{4}{25}x^2\sqrt{x} + C}$$

4.



$$A(R)=\int_0^1 \sqrt{x}-x^2\,dx$$

$\boxed{\sqrt{x}>x^2 \text{ for } 0\le x\le 1}$

$$=\left[\frac{2}{3}x^{3/2}-\frac{1}{3}x^3\right]_0^1$$

$$=\left(\frac{2}{3}\cdot 1^{3/2}-\frac{1}{3}\cdot 1^3\right)-0$$

$$=\frac{2}{3}-\frac{1}{3}=\underline{\underline{\frac{1}{3}}}$$

$x^2=\sqrt{x}$

$x^2-\sqrt{x}=0$

$\sqrt{x}\,(x^{3/2}-1)=0$

$\underline{x=0}$ eller $x^{3/2}=1$

$\underline{x=1}$

5.

a) $$\int \frac{1}{1-\sqrt{x}}\,dx = \int \frac{1}{u}\cdot\frac{du}{-\frac{1}{2\sqrt{x}}}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}$$

$$\boxed{\begin{aligned}u&=1-\sqrt{x}\\ du&=-\frac{1}{2\sqrt{x}}\,dx\end{aligned}}$$

$\sqrt{x}=1-u$

$$=\int \frac{1}{u}\,\frac{2\sqrt{x}}{-1}\,du=-2\int\frac{\sqrt{x}}{u}\,du=-2\int\frac{1-u}{u}\,du$$

$$=-2\int \frac{1}{u}-1\,du=-2(\ln|u|-u)+C$$

$$=-2\ln|u|+2u+C$$

$$=\underline{-2\ln|1-\sqrt{x}|+2(1-\sqrt{x})+C}$$

b) $$\int_0^b \frac{1}{1-\sqrt{x}}\,dx=\left[-2\ln|1-\sqrt{x}|+2(1-\sqrt{x})\right]_0^b$$

$$=\left(-2\ln(1-\sqrt{b})+2(1-\sqrt{b})\right)-\left(-2\ln(1)+2\cdot(1-0)\right)$$

$$=-2\ln(1-\sqrt{b})+\cancel{2}-2\sqrt{b}-\cancel{2}=\underline{-2\ln(1-\sqrt{b})-2\sqrt{b}}$$

$b\in(0,1)$

$\Downarrow$

$\sqrt{b}\in(0,1)$

$1-\sqrt{b}\in(0,1)$

$\ln(1-\sqrt{b})<0$ er definert

c) $b \to 1$ gir $-2 \cdot \ln(1-\sqrt{b}) - 2\sqrt{b} \to \infty$

siden $\sqrt{b} \to 1$

$1-\sqrt{b} \to 0$

$\ln(1-\sqrt{b}) \to -\infty$

Altså har vi: $\int_0^1 \frac{1}{1-\sqrt{x}}\,dx = \lim \int_0^b \frac{1}{1-\sqrt{x}}\,dx = \underline{\underline{\infty}}$

(integralet eksisterer altså ikke)

6.

a)
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & -1 & -7 & 4 \\ 1 & 2 & 5 & h \end{array}\right) \begin{matrix} -1 \\ -1 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & -2 & -8 & -8 \\ 0 & 1 & 4 & h-12 \end{array}\right) \updownarrow$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 4 & h-12 \\ 0 & -2 & -8 & -8 \end{array}\right)]2 \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 4 & h-12 \\ 0 & 0 & 0 & 2h-32 \end{array}\right)$$

$\underline{2h-32=0}$

$h=16$

uendelig mange løsn.

$$\left(\begin{array}{ccc|c} 1 & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\underline{2h-32 \neq 0}$

$h \neq 16$

ingen løsninger

$$\left(\begin{array}{ccc|c} 1 & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot \\ 0 & 0 & 0 & 2h-32 \end{array}\right)$$

Konkl:

systemet inkonsistent for $\underline{\underline{h \neq 16}}$

b) Konsistent: $h=16$

Trappeform:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x+y+z=12$

$y+4z=4$

(z fri)

$x = 12-y-z = 12-(4-4z)-z = 8+3z$

$y = 4-4z$

$(x,y,z) = \underline{\underline{(8+3z, 4-4z, z)}}$

der z er fri

7. $$A = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ a & 0 & 1 & 0 \\ 0 & a & 0 & 1 \end{pmatrix}$$

a) $$|A| = \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ a & 0 & 1 & 0 \\ 0 & a & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{vmatrix} + a \cdot \begin{vmatrix} 0 & a & 0 \\ 1 & 0 & a \\ a & 0 & 1 \end{vmatrix}$$

$$= 1 \cdot \left(+1 \cdot (1-a^2)\right) + a \cdot \left(-a(1-a^2)\right)$$

$$= (1-a^2) - a^2 \cdot (1-a^2) = (1-a^2) \cdot (1-a^2)$$

$$= (1+a)(1-a) \cdot (1+a)(1-a) = \underline{\underline{(1+a)^2(1-a)^2}}$$

b) $|A| = 0$: $\quad (1+a)^2 \cdot (1-a)^2 = 0$

$\underline{\underline{a = -1}}$, $\underline{\underline{a = 1}}$ (to løsn. med multiplisitet to)

c) $A \cdot \underline{x} = \underline{0}$ har én løsn $\iff |A| \neq 0$

$\underline{\underline{a \neq -1,\ a \neq 1}}$

d) Uendelig mange løsn: $a = -1, a = 1$ (systemet er konsistent siden siste kolonne er null, og ikke kan ha pivot-pos.)

$a = 1$: $$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} x_1 + x_3 = 0 \\ x_2 + x_4 = 0 \\ x_3, x_4 \text{ fri} \end{array}\right\} \implies \begin{array}{l} x_1 = -x_3 \\ x_2 = -x_4 \\ x_3, x_4 \text{ fri} \end{array}$$

$$\underline{x} = (x_1, x_2, x_3, x_4) = \underline{\underline{(-x_3, -x_4, x_3, x_4)}}$$

der x_3, x_4 fri

$\underline{a=-1}$:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

x_3, x_4 fri

$x_1 - x_3 = 0$

$x_2 - x_4 = 0$

x_3, x_4 fri

$$\Rightarrow \underline{X} = (x_1, x_2, x_3, x_4) = \underline{\underline{(x_3, x_4, x_3, x_4)}}$$

der x_3, x_4 er fri

$x_1 = x_3$

$x_2 = x_4$

$\underline{8.}$

$$\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \end{array}\right)$$

viser en mulig trappeform (den eneste mulige, faktisk)

- konsistent (ikke pivot-pos. i siste kolonne)
- ingen fri var.

⇓

<u>én løsning</u>

$\underline{9.}$

a)

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & -1 & 1 \\ 2 & -1 & -5 & 2 & -1 \\ 5 & 5 & 7 & -1 & 2 \end{array}\right) \begin{array}{l} \\ ]\,{-2} \\ ]\,{-5} \end{array}$$

$x_1 + 2x_2 + 4x_3 - x_4 = 1$

$2x_1 - x_2 - 5x_3 + 2x_4 = -1$

$5x_1 + 5x_2 + 7x_3 - x_4 = 2$

b)

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & -1 & 1 \\ 0 & -5 & -13 & 4 & -3 \\ 0 & -5 & -13 & 4 & -3 \end{array}\right)]\,{-1} \to \left(\begin{array}{cccc|c} 1 & 2 & 4 & -1 & 1 \\ 0 & -5 & -13 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

<u>uendelig mange løsn.</u> og

og to frie variable $\underline{x_3, x_4}$

Løsn:

$$x_1 + 2x_2 + 4x_3 - x_4 = 1$$
$$-5x_2 - 13x_3 + 4x_4 = -3$$

$$-5x_2 = -3 + 13x_3 - 4x_4$$
$$x_2 = 3/5 - 13/5\, x_3 + 4/5\, x_4$$

$$x_1 = 1 - 2x_2 - 4x_3 + x_4$$
$$= 1 - 2\left(3/5 - 13/5\, x_3 + 4/5\, x_4\right) - 4x_3 + x_4$$
$$= -1/5 + \frac{6}{5}x_3 - 3/5\, x_4$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1/5 + 6/5\, x_3 - 3/5\, x_4 \\ 3/5 - 13/5\, x_3 + 4/5\, x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

10.

a) $$|A| = \begin{vmatrix} 1 & a & a \\ 0 & 2 & 0 \\ a & a^2 & 1 \end{vmatrix} = +2\cdot(1-a^2) = 2(1-a^2)$$
$$= 2(1-a)(1+a)$$

b) $\underline{a=2}$: $|A| = 2\cdot(-1)\cdot 3 = \underline{-6} \neq 0$

$$A^{-1} = \frac{1}{-6}\cdot\begin{pmatrix} 2 & 0 & -2a \\ a^3-a & 1-a^2 & a^3-a^2 \\ -2a & 0 & 2 \end{pmatrix}^T = -\frac{1}{6}\begin{pmatrix} 2 & 0 & -4 \\ 6 & -3 & 2 \\ -4 & 0 & 2 \end{pmatrix}^T$$

$$A^{-1} = -\frac{1}{6}\begin{pmatrix} 2 & 6 & -4 \\ 0 & -3 & 0 \\ -4 & 2 & 2 \end{pmatrix}$$

b) $A\underline{x} = \underline{b}$ $\Downarrow$ $|A| \neq 0$

$A^{-1}A\underline{x} = A^{-1}\underline{b}$

$$\underline{x} = A^{-1}\underline{b} = -\frac{1}{6}\begin{pmatrix} 2 & 6 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} +12 \\ -12 \\ -12 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}}}$$

c) $|A| = 0$

$\Updownarrow$

$2(1-a)(1+a) = 0$

$\underline{a=1}, \underline{a=-1}$

$\underline{a \neq 1, -1}$: $|A| \neq 0$, én løsning

$\underline{a=1}$:
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 1 & 6 \end{array}\right) \begin{matrix} \\ \\ -1 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

uendelig mange løsn.

$\underline{a=-1}$:
$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 6 \\ 0 & 2 & 0 & -2 \\ -1 & 1 & 1 & 6 \end{array}\right) \begin{matrix} \\ \\ 1 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & 6 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right)$$

ingen løsn.

Konkl:
i) ingen løsn: $a = -1$
ii) én løsn: $a \neq 1, -1$
iii) uendelig mange løsn: $\underline{\underline{a = 1}}$

d) $\underline{a=1}$:

$x + y + z = 6$ $\quad\quad$ $x = 6 - y - z = \underline{5 - z}$

$2y = 2$ $\quad\quad$ $\underline{y = 1}$

$(z \text{ fri})$

$(x, y, z) = \underline{\underline{(5-z, 1, z)}}$ z fri

e) $\underline{a \neq 1, -1}$: $|A| = 2(1-a)(1+a) \neq 0$

$$x = \frac{|A_1(\underline{b})|}{|A|} = \frac{2a^4-2a^3-12a+12}{2(1-a)(1+a)}$$

$$= \frac{\cancel{2}\,\overset{-1}{\cancel{(a-1)}}\,(a^3\overset{+a^2}{}-6)}{\cancel{2}\,\cancel{(1-a)}\,(1+a)}$$

$$= -\frac{a^3+a^2-6}{a+1}$$

$$|A_1(b)| = \begin{vmatrix} 6 & a & a \\ 2a & 2 & 0 \\ 6 & a^2 & 1 \end{vmatrix}$$

$$= a\cdot(2a^3-12)+1\cdot(12-2a^2)$$

$$= 2a^4-2a^2-12a+12$$

$$= 2(a^4-a^2-6a+6)$$

$$= 2(a-1)(a^3+a^2-6)$$

$$y = \frac{|A_2(\underline{b})|}{|A|} = \frac{2a\cancel{(1-a)}\cancel{(1+a)}}{\cancel{2}\cancel{(1-a)}\cancel{(1+a)}}$$

$$= a$$

$$|A_2(\underline{b})| = \begin{vmatrix} 1 & 6 & a \\ 0 & 2a & 0 \\ a & 6 & 1 \end{vmatrix}$$

$$= 2a\cdot(1-a^2) = 2a(1-a)(1+a)$$

$$z = \frac{|A_3(b)|}{|A|} = \frac{\overset{6}{\cancel{12}}\cancel{(1-a)}}{\cancel{2}\cancel{(1-a)}(1+a)}$$

$$= \frac{6}{a+1}$$

$$|A_3(\underline{b})| = \begin{vmatrix} 1 & a & 6 \\ 0 & 2 & 2a \\ a & a^2 & 6 \end{vmatrix}$$

$$= 1\cdot(12-2a^3)+a(2a^2-12)$$

$$= 12-\cancel{2a^3}+\cancel{2a^3}-12a$$

$$= 12-12a = 12(1-a)$$

$$a^4-a^2-6a+6 : a-1 = a^3+a^2-6$$

$-(a^4-a^3)$

a^3-a^2-6a+6

$-(a^3-a^2)$

$-6a+6$

$-(-6a+6)$

0

Konklusjon: for $a \neq 1, -1$ er

$$(x, y, z) = \left(-\frac{a^3+a^2-6}{a+1},\ a,\ \frac{6}{a+1}\right)$$