Løsning: Oppgaveark 3

1. R = antall riktige svar $\qquad R \sim \text{Binom}(n,p)$ med $n=15$, $p=1/4$

a) $15-R$ = antall gale svar

X = antall poeng

$\Downarrow$

$$X = R\cdot 3 + (15-R)\cdot(-1)$$
$$= 3R - 15 + R = \underline{4R-15}$$

b) $$E(X) = E(4R-15) = 4E(R) - 15$$
$$= 4\cdot np - 15 = 4\cdot 15\cdot 1/4 - 15 = \underline{\underline{0}}$$

$$Var(X) = Var(4R-15) = 4^2\cdot Var(R)$$
$$= 4^2\cdot np(1-p) = \cancel{16}\cdot 15\cdot\frac{1}{\cancel{4}}\cdot\frac{3}{\cancel{4}} = \underline{45}$$

$$\Rightarrow \sigma_X = \sqrt{Var X} = \underline{\underline{\sqrt{45}}} \approx \underline{\underline{6.71}}$$

c) $$P(X \geq 37) = P(4R-15 \geq 37)$$
$$= P(4R \geq 37+15 = 52) = P(R \geq 52/4 = 13)$$
$$= P(R=13) + P(R=14) + P(R=15)$$

$$p(R=13) = \binom{15}{13} \cdot (1/4)^{13} \cdot (3/4)^{2}$$

$$= \frac{15 \cdot 14}{2} \cdot \frac{9}{4^{15}} = 945 \cdot (1/4)^{15}$$

$$p(R=14) = \binom{15}{14} \cdot (1/4)^{14} \cdot (3/4) = 45 \cdot (1/4)^{15}$$

$$p(R=15) = \binom{15}{15} \cdot (1/4)^{15} = (1/4)^{15}$$

$$\Downarrow$$

$$p(X \geq 37) = (945 + 45 + 1) \cdot (1/4)^{15} = 991 \cdot (1/4)^{15}$$

$$\simeq \underline{\underline{0{,}00000092}} = \underline{0{,}00009\%}$$

$$p(X<9) = p(4R-15<9) = p(4R<24)$$

$$= p(R<6) = p(R \leq 5) = p(0) + \ldots + p(5)$$

$$p(0) = \binom{15}{0} \cdot (3/4)^{15} = (3/4)^{15} \simeq 0{,}0134$$

$$p(1) = \binom{15}{1} \cdot (1/4)(3/4)^{14} \simeq 0{,}0668$$

$$p(2) = \binom{15}{2} (1/4)^2 (3/4)^{13} \simeq 0{,}1559$$

$$p(3) = \binom{15}{3} (1/4)^3 (3/4)^{12} \simeq 0{,}2252$$

$$p(4) = \binom{15}{4} (1/4)^4 (3/4)^{11} \simeq 0{,}2252$$

$$p(5) = \binom{15}{5} (1/4)^5 (3/4)^{10} \simeq \underline{0{,}1651}$$

$$p(X<9) = p(R \leq 5) \simeq \underline{\underline{0{,}852}} = \underline{85{,}2\%}$$

d) Ny forutsetning: $p = 1/3$

$R \sim \text{Binom}(n=15, p=1/5)$

$Y = 4R - 15$

$E(Y) = 4\,E(R) - 15 = 4\cdot(15\cdot 1/3) - 15 = \underline{\underline{5}}$

$Var(Y) = 4^2 \cdot Var(R) = 16\cdot(15\cdot 1/3 \cdot 2/3)$

$= 16\cdot \frac{30}{9} = \frac{160}{3} \Rightarrow \sigma_Y = \underline{\underline{\sqrt{\frac{160}{3}}}}$

$\approx \underline{7{,}30}$

e) $p(Y \geq 37) = p(R \geq 13)$

$= p(R=13) + p(R=14) + p(R=15)$

$= \binom{15}{13}\cdot(1/3)^{13}\cdot(2/3)^2 + \binom{15}{14}(1/3)^{14}(2/3) + \binom{15}{15}(1/3)^{15}$

$= 420\cdot(1/3)^{15} + 30(1/3)^{15} + (1/3)^{15} = 451\cdot(1/3)^{15}$

$\approx \underline{\underline{0{,}000031}}$

$p(Y<9) = p(R<6) = p(0) + \ldots + p(5)$

$p(0) = (2/3)^{15} \approx 0{,}0029$

$p(1) = 15(1/3)(2/3)^{14} \approx 0{,}0171$

$p(2) = \binom{15}{2}(1/3)^2(2/3)^{13} \approx 0{,}0599$

$p(3) = \binom{15}{3}(1/3)^3(2/3)^{12} \approx 0{,}1299$

$p(4) = \binom{15}{4}(1/3)^4(2/3)^{11} \approx 0{,}1948$

$p(5) = \binom{15}{5}(1/3)^5\cdot(2/3)^{10} \approx 0{,}2143$

$p(Y<9) \approx \underline{\underline{0{,}618}} \approx \underline{61{,}8\%}$

f) Ny forutsetning: $p = 1/2$

$R \sim \text{Binom}(n=15, p=1/2)$

$Z = 4R - 15$

$E(Z) = 4 \cdot E(R) - 15 = 4 \cdot (15 \cdot 1/2) - 15 = \underline{\underline{15}}$

$Var(Z) = 4^2 \, Var(R) = 16 \cdot (15 \cdot 1/2 \cdot 1/2) = \underline{60}$

$\Rightarrow \sigma_Z = \underline{\underline{\sqrt{60}}} \simeq \underline{\underline{7,75}}$

g) $p(Z \geq 37) = p(R \geq 13)$

$p(13) = \binom{15}{13} (1/2)^{13} \cdot (1/2)^2 \simeq 0{,}00320$

$p(14) = \quad \simeq 0{,}00046$

$p(15) = \quad \simeq 0{,}00003$

$p(Z \geq 37) = p(R \geq 13) \simeq \underline{\underline{0.00369}}$

$p(Z < 9) = p(R < 6)$

$p(0) = (1/2)^{15}$

$p(1) = 15 \cdot (1/2)^{15}$

$p(2) = 105 \cdot (1/2)^{15}$

$p(3) = 455 \cdot (1/2)^{15}$

$p(4) = 1365 \cdot (1/2)^{15}$

$p(5) = 3003 \cdot (1/2)^{15}$

$p(Z < 9) = p(R < 6)$

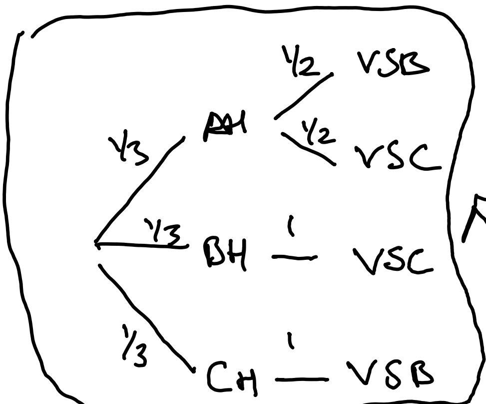
$= (1 + 15 + 105 + 455 + 1365 + 3003) \cdot (1/2)^{15}$

$\simeq \underline{\underline{0{,}151}} \simeq \underline{\underline{15.1\%}}$

2.

AH : Fange A henrettes
BH : " B "
CH : " C "

$p(AH) = p(BH) = p(CH) = \underline{1/3}$



VSB: Vokter svarer at B settes fri

VSC: Vokter svarer at C settes fri

$p(VSB|AH) = 1/2$

$p(VSC|AH) = 1/2$

$p(VSB|BH) = 0$

$p(VSC|BH) = 1$

$p(VSB|CH) = 1$

$p(VSC|CH) = 0$

$$p(AH|VSB) = \frac{p(AH \cap VSB)}{p(VSB)}$$

$$= \frac{1/3 \cdot 1/2}{1/3 \cdot 1/2 + 1/3 \cdot 1} = \frac{1/6}{1/6 + 2/6}$$

$$= \frac{1/6}{3/6} = \underline{\underline{1/3}} = p(AH)$$

Argumentasjonen er <u>ikke</u> riktig; sannsynligheten for at A henrettes er den samme.

3. X = antall kast til man får K

U = utbetaling

$\underline{U = 2^X}$:

X	1	2	3	4	...
U	2	4	8	16	...

a) $E(U) = u_1 \cdot p(u_1) + u_2 \cdot p(u_2) + \cdots$

$= 2 \cdot 1/2 + 4 \cdot 1/4 + 8 \cdot 1/8 + \ldots$

$= 1 + 1 + 1 + \ldots = \underline{\infty}$

Forventet (gjennomsnittlig) utbetaling er uendelig stor, man burde være villig til å betale en uendelig pris for å delta.

b)

X	1	2	3	4	5	6	7	8	9	10	11	12
U	2	4	8	16	32	64	128	256	512	1024	2048	4096

13	14	15	...
8192	0	0	...

$E(U) = 2 \cdot 1/2 + 4 \cdot 1/4 + \ldots + 8192 \cdot \frac{1}{8192}$

$= \underbrace{1 + 1 + 1 + \ldots + 1}_{13 \text{ ganger}} = \underline{\underline{13}}$

Du burde være villig til å betale $\underline{\underline{13 \text{kr}}}$.

4. $X(S) = \{1, 2, 3, 4, \ldots\}$

x	1	2	3	4	5	...
$p(x)$	$1/2$	$1/4$	$1/8$	$1/16$	$1/32$	...
$F(x)$	$1/2$	$3/4$	$7/8$	$15/16$	$31/32$	...

5. a) $p(x>2) = 1 - p(x \le 2)$

$= 1 - F(2) = 1 - 3/4 = \underline{\underline{1/4}}$

b) $p(x \ge 2) = 1 - p(x<2) = 1 - p(x \le 1)$

$= 1 - F(1) = 1 - 1/2 = \underline{\underline{1/2}}$

c) $p(3 < x \le 5) = F(5) - F(3)$

$= 31/32 - 7/8 = \frac{31}{32} - \frac{7 \cdot 4}{8 \cdot 4} = \underline{\underline{3/32}}$

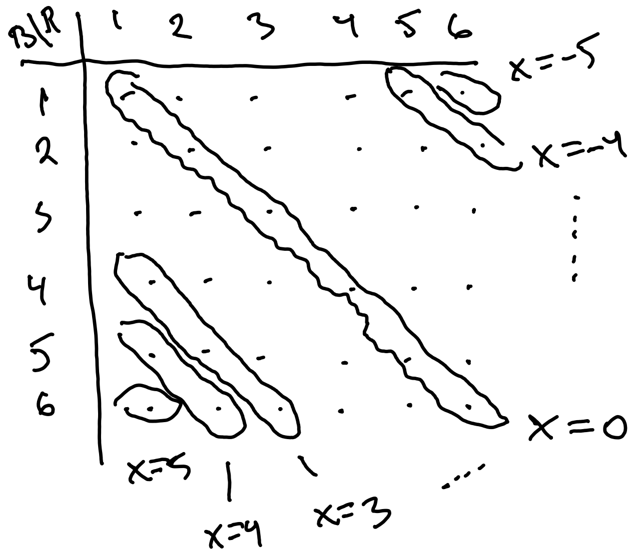
d) $p(3 \le x \le 5) = F(5) - F(2)$

$= \frac{31}{32} - \frac{3 \cdot 8}{4 \cdot 8} = \underline{\underline{\frac{7}{32}}}$

e) $p(x>5) = 1 - p(x \le 5) = 1 - F(5)$

$= 1 - 31/32 = \underline{\underline{1/32}}$

6.



X = blå - rød terning

$X(S) = \{-5, -4, \dots, 4, 5\}$

x	$p(x)$	$F(x)$
-5	1/36	1/36
-4	2/36	3/36
-3	3/36	6/36
-2	4/36	10/36
-1	5/36	15/36
0	6/36	21/36
1	5/36	26/36
2	4/36	30/36
3	3/36	33/36
4	2/36	35/36
5	1/36	36/36

7.

a) $E(X) = (-5)\cdot 1/36 + (-4)\cdot 2/36 + \dots$

$\dots + 5\cdot 1/36 = \dfrac{-5\cdot 1 + (-4)\cdot 2 + (-3)\cdot 3 + (-2)\cdot 4 + \dots + 5\cdot 1}{36} = 0$

b) $E(X^2) = (-5)^2 \cdot 1/36 + (-4)^2 \cdot 2/36 + \dots$

$= 2\cdot \dfrac{5^2\cdot 1 + 4^2\cdot 2 + 3^2\cdot 3 + 2^2\cdot 4 + 1^2\cdot 5}{36} = \dfrac{210}{36} \simeq 5.83$

$(-5)^2 \cdot 1/36 = 5^2 \cdot 1/36$ etc

c) $Var(X) = E(X^2) - E(X)^2 = \frac{210}{36} - 0^2 = \underline{\underline{\frac{210}{36}}}$

$\simeq \underline{5.83}$

d) $E(X+1) = E(X)+1 = 0+1 = \underline{\underline{1}}$

e) $Var(X+1) = Var(X) = \underline{\underline{\frac{210}{36}}} \simeq \underline{\underline{5.83}}$

$\underline{8.}$ $E(X) = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + \dots$

$$\left\{\begin{array}{llr}
= & \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots & \left(= \frac{1}{2}\cdot\frac{1}{1-1/2} = 1\right) \\
+ & \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots & \left(= \frac{1}{4}\cdot\frac{1}{1-1/2} = \frac{1}{2}\right) \\
+ & \frac{1}{8}+\frac{1}{16}+\dots & \left(= \frac{1}{8}\cdot\frac{1}{1-1/2} = \frac{1}{4}\right) \\
+ & & + \\
\vdots & & \vdots
\end{array}\right.$$

$= 1 + 1/2 + 1/4 + \dots = 1\cdot\frac{1}{1-1/2} = \underline{\underline{2}}$

9. X = antall kast før vi slår K

U = utbetaling ($= 2^X$ når $X \leq 13$)

I = 13 kr

N = nettogevinst $= U - I$

$$E(N) = E(U) - E(I) = 13 - 13 = \underline{\underline{0}}$$

(se opps. 3b)

$$Var(N) = Var(U - 13) = Var(U)$$

$\underline{Var(U)}$: $E(U^2) = 2^2 \cdot \frac{1}{2} + 4^2 \cdot \frac{1}{4} + 8^2 \cdot \frac{1}{8} + \ldots$

$\ldots + 8192^2 \cdot \frac{1}{8192}$

$$= \underbrace{2 + 4 + 8 + \ldots + 8192}_{13 \text{ ledd}} = 2 \cdot \frac{2^{13} - 1}{2 - 1}$$

$$= 2(2^{13} - 1) = \underline{16.382}$$

$$Var(U) = E(U^2) - E(U)^2 = 16.382 - 13^2$$

$$= 16.213$$

$$Var(N) = Var(U) = 16.213$$

$$\sigma_N = \sqrt{16.213} \simeq \underline{\underline{127.3}}$$

10.

$$E(aX+b) = (ax_1+b)\,p(x_1) + (ax_2+b)\,p(x_2) + \dots$$

$$= a\cdot\underbrace{(x_1 p(x_1) + x_2 p(x_2) + \dots)}_{E(X)} + b\cdot\underbrace{(p(x_1)+p(x_2)+\dots)}_{1}$$

$$= a\cdot E(X) + b$$

11.

$$Var(X) = E[(X-\mu)^2] = E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$