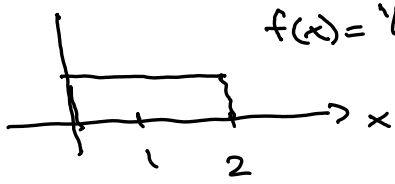
Løsning: Oppgavearkt 5

1.



$f(x) = 1/2$, $0 \le x \le 2$

a) $E(x) = \int_0^2 x \cdot f(x)\,dx = \int_0^2 x \cdot \frac{1}{2}\,dx = \left[\frac{1}{2}\cdot\frac{1}{2}x^2\right]_0^2$

$= \frac{1}{4}\cdot 2^2 - 0 = \underline{\underline{1}}$

b) $E(x^2) = \int_0^2 x^2 \cdot \frac{1}{2}\,dx = \left[\frac{1}{2}\cdot\frac{1}{3}x^3\right]_0^2$

$= \frac{1}{6}\cdot 2^3 - 0 = 8/6 = 4/3$

$\Rightarrow Var(x) = E(x^2) - \mu^2 = 4/3 - 1^2 = \underline{\underline{1/3}}$

c) $P(0.7 \le x \le 1.1) = \int_{0.7}^{1.1} \frac{1}{2}\,dx = \left[\frac{1}{2}x\right]_{0.7}^{1.1}$

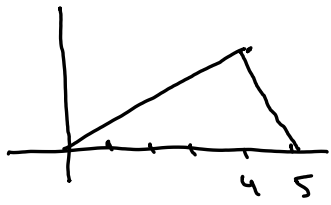
$= \frac{1}{2}(1.1 - 0.7) = \frac{1}{2}\,0.4 = \underline{\underline{0.2}}$

d) $F(x) = \int_0^x f(x)\,dx = \int_0^x \frac{1}{2}\,dx = \left[\frac{1}{2}x\right]_0^x$

$= \underline{\underline{\frac{1}{2}x}}$, $0 \le x \le 2$

($F(x) = 0$ for $x < 0$, $F(x) = 1$ for $x > 2$)

2.



$$f(x)=\begin{cases}0.1x, & 0\le x\le 4\\ 2-0.4x, & 4\le x\le 5\\ (0, & \text{else}\end{cases}$$

a) $E(x)=\int_0^5 x f(x)\,dx=\int_0^4 x\cdot 0.1x\,dx+\int_4^5 x(2-0.4x)\,dx$

$=\left[0.1\cdot\frac{1}{3}x^3\right]_0^4+\left[x^2-0.4\cdot\frac{1}{3}x^3\right]_4^5$

$=\frac{1}{30}\cdot 4^3-0+\left(25-\frac{4}{30}\cdot 5^3\right)-\left(16-\frac{4}{30}4^3\right)$

$=\frac{64}{30}+9-\frac{4\cdot 125}{30}+\frac{4\cdot 64}{30}=\frac{9\cdot 30+64-500+256}{30}$

$=\frac{90}{30}=\underline{\underline{3}}$

b) $E(x^2)=\int_0^5 x^2 f(x)\,dx=\int_0^4 x^2\cdot 0.1x\,dx+\int_4^5 x^2(2-0.4x)\,dx$

$=\left(\frac{1}{40}\cdot 4^4-0\right)+\left[\frac{2}{3}x^3-\frac{1}{10}x^4\right]_4^5=\frac{64}{10}+\left(\frac{2}{3}\cdot 5^3-\frac{1}{10}\cdot 5^4\right)$

$-\left(\frac{2}{3}\cdot 4^3-\frac{1}{10}\cdot 4^4\right)=\frac{64}{10}+\frac{250}{3}-\frac{125}{2}-\frac{128}{3}+\frac{128}{5}$

$=\frac{64\cdot 3+250\cdot 10-125\cdot 15-128\cdot 10+128\cdot 6}{30}=\frac{61}{6}$

$\Rightarrow \mathrm{Var}(x)=\frac{61}{6}-3^2=\frac{61}{6}-\frac{9\cdot 6}{6}=\underline{\underline{\frac{7}{6}}}$

c) $P(2 \le x \le 3) = \int_2^3 0.1x\,dx = \left[0.05x^2\right]_2^3$

$= 0.05 \cdot (9-4) = \underline{0.25}$

d) $F(x)$:

$0 \le x \le 4$: $F(x) = \int_0^x 0.1x\,dx = \frac{0.1}{2}x^2 \Big|_0^x$

$= \underline{0.05x^2} = \underline{x^2/20}$

$4 \le x < 5$: $F(x) = \int_0^4 0.1x\,dx + \int_4^x 2 - 0.4x\,dx$

$= 0.8 + \left[2x - 0.2x^2\right]_4^x$

$= 0.8 + (2x - 0.2x^2) - (8 - 3.2)$

$= \underline{-x^2/5 + 2x - 4}$

$$F(x) = \begin{cases} 0 & , x < 0 \\ x^2/20 & , 0 \le x \le 4 \\ -x^2/5 + 2x - 4 & , 4 \le x \le 5 \\ 1 & , x > 5 \end{cases}$$

3. $f(x) = F'(x) = 3x^2/8$, $0 \le x \le 2$

$$E(x) = \int_0^2 x \cdot 3x^2/8 \, dx = \left[\frac{3}{8} \cdot \frac{1}{4} x^4 \right]_0^2$$

$$= \frac{3}{24} \cdot 2^4 - 0 = \frac{3 \cdot 16}{24} = 3/2$$

4. $f(x) = ax^2$, $0 \le x \le 5$

$$\int_0^5 f(x)\,dx = 1$$

$$\int_0^5 ax^2 \, dx = a \cdot \frac{1}{3} x^3 \Big|_0^5 = a \cdot \frac{125}{3} = 1$$

$$a = \frac{3}{125}$$

$$E(x) = \int_0^5 x \cdot \frac{3}{125} x^2 \, dx$$

$$= \left[\frac{3}{125} \cdot \frac{1}{4} x^4 \right]_0^5 = \frac{3}{500} \cdot 5^4 - 0 = \frac{15}{4} = 3.75$$

5. $X \sim N(\mu=950, \sigma=120)$

$\Downarrow$

$Z = \frac{X-950}{120} \sim N(0,1)$

$p(X>1100) = 1 - p(X \leq 1100)$

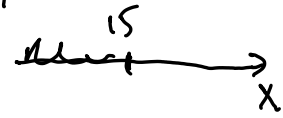
$= 1 - p\left(Z \leq \frac{1100-950}{120} = 1.25\right)$

$= 1 - \Phi(1.25) \cong \underline{0.106} = \underline{10.6\%}$

6. $X \sim N(\mu=20, \sigma=5) \Rightarrow Z = \frac{X-20}{5} \sim N(0,1)$

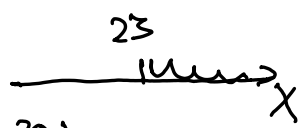
a) $p(X<15) = p\left(Z \leq \frac{15-20}{5} = -1\right)$

$= \Phi(-1) \simeq \underline{0.159}$



b) $p(X>23) = p\left(Z > \frac{23-20}{5} = 0.6\right)$

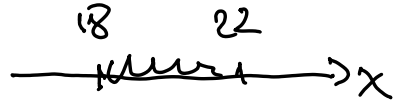
$= 1 - \Phi(0.6) \simeq \underline{0.274}$



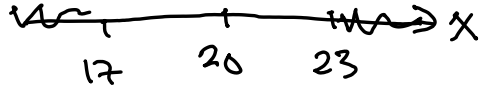
c) $p(18 \leq X \leq 22) = p\left(\frac{18-20}{5} \leq Z \leq \frac{22-20}{5}\right)$

$= p(-0.4 \leq Z \leq 0.4) = \Phi(0.4) - \Phi(-0.4)$

$\simeq \underline{0.311}$



d) $p(|X-20|>3)$



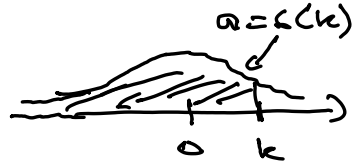
$= p(X>23) + p(X<17)$

$= p\left(Z > \frac{23-20}{5}\right) + p\left(Z < \frac{17-20}{5}\right)$

$= p(Z>0.6) + p(Z<-0.6)$

$= 2 \cdot \Phi(-0.6) \simeq 0.549$

7.



a) $p(Z>k) = 1 - p(Z \leq k)$

$= 1 - a$

b) $p(Z<-k) = p(Z \geq k)$

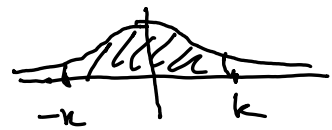
$= 1-a$

pga symmetri

c) $p(-k \leq Z \leq k)$

$= p(Z \leq k) - p(Z \leq -k)$

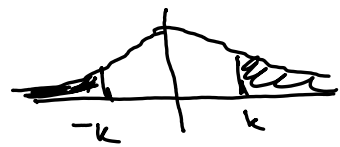
$= a - (1-a) = 2a-1$



d) $p(|X|>k)$

$= 1 - p(-k \leq Z \leq k) = 1 - (2a-1)$

$= 2 - 2a$



8.

a) $p(Z<k) = \Phi(k) = 0.95$

$k = \Phi^{-1}(0.95) \simeq 1.645$

b) $p(-k<Z<k) = 0.90$

$\Downarrow$

$p(|Z| \geq k) = 1 - 0.90 = 0.10$

$p(Z>k) = \frac{0.10}{2} = 0.05$

$\Rightarrow p(Z \leq k) = 1 - 0.05 = 0.95$

$k = \Phi^{-1}(0.95) \simeq 1.645$

c) $p(Z>k) = 0.025$

$p(Z<k) = 1 - 0.025 = 0.975$

$\Phi(k) = 0.975 \Rightarrow k = \Phi^{-1}(0.975) \simeq 1.96$

d) $p(|Z|>k) = 0.05$

$\Downarrow$

$p(Z>k) = p(Z<-k) = \frac{0.05}{2}$

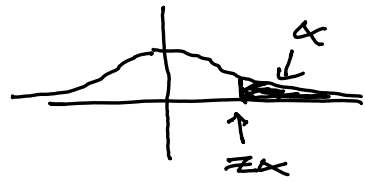
$= 0.025$

$p(Z \leq k) = \Phi(k) = 1 - 0.025 = 0.975$

$\Rightarrow k = \Phi^{-1}(0.975) \simeq 1.96$

9. Defn: z_α er z-verdien slik at

$$P(Z > z_\alpha) = \alpha$$
$$\Updownarrow$$
$$P(Z \leq z_\alpha) = 1 - \alpha$$



$$z_\alpha = G^{-1}(1-\alpha)$$

$\alpha = 0.08$: $z_\alpha = G^{-1}(0.92) \approx 1.405$

$\alpha/2 = 0.04$: $z_{\alpha/2} = G^{-1}(0.96) \approx 1.751$

Husk: For å regne ut på kalkulator:

z [Z⇄P] gir $G(z)$

p [INV] [Z⇄P] gir $G^{-1}(p)$

10.

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx = \int_{-\infty}^{\infty} ax f(x) + b f(x)\,dx$$

$$= a \cdot \underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{E(X)} + b \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1} = aE(X) + b \cdot 1$$

$$= aE(X) + b$$