Løsning: Oppgaveark 6

1.

	$Y=1$	2	
$X=1$	0.4	0.3	0.7
2	0.2	0.1	0.3
	0.6	0.4	1.

Marginale fordelinger:

$p_X(1) = 0.7$ $\quad p_Y(1) = 0.6$

$p_X(2) = 0.3$ $\quad p_Y(2) = 0.4$

a) $E(X) = 1 \cdot 0.7 + 2 \cdot 0.3 = \underline{1.3}$

b) $E(X^2) = 1^2 \cdot 0.7 + 2^2 \cdot 0.3 = \underline{1.9}$

$\Rightarrow Var(X) = 1.9 - 1.3^2 = \underline{0.21}$

c) $E(Y) = 1 \cdot 0.6 + 2 \cdot 0.4 = \underline{1.4}$

d) $E(Y^2) = 1^2 \cdot 0.6 + 2^2 \cdot 0.4 = \underline{2.2}$

$\Rightarrow Var(Y) = 2.2 - 1.4^2 = \underline{0.24}$

e) $E(XY) = 1 \cdot 1 \cdot 0.4 + 1 \cdot 2 \cdot 0.3 + 2 \cdot 1 \cdot 0.2 + 2 \cdot 2 \cdot 0.1 = \underline{1.80}$

f) $E(X) \cdot E(Y) = 1.3 \cdot 1.4 = \underline{1.82}$

g) $Cov(X,Y) = E(XY) - E(X)E(Y) = 1.80 - 1.82 = \underline{-0.02}$

h) $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \dfrac{-0.02}{\sqrt{0.21} \cdot \sqrt{0.24}} \simeq \underline{-0.089}$

2.

	Y=1	2	3	4	5	6	
X=1	1/36	0	0	0	0	0	1/36
2	2/36	1/36	0	0	0	0	3/36
3	2/36	2/36	1/36	0	0	0	5/36
4	2/36	2/36	2/36	1/36	0	0	7/36
5	2/36	2/36	2/36	2/36	1/36	0	9/36
6	2/36	2/36	2/36	2/36	2/36	1/36	11/36
	11/36	9/36	7/36	5/36	3/36	1/36	1

a) $E(X) = 1\cdot\frac{1}{36} + 2\cdot\frac{3}{36} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{9}{36} + 6\cdot\frac{11}{36} = \frac{161}{36} \simeq \underline{4.47}$

b) $E(X^2) = 1^2\cdot\frac{1}{36} + 2^2\cdot 3/36 + 3^2\cdot\frac{5}{36} + 4^2\cdot\frac{7}{36} + 5^2\cdot\frac{9}{36} + 6^2\cdot\frac{11}{36}$

$= \frac{791}{36} \simeq 21.97 \Rightarrow Var(X) = \frac{791}{36} - \left(\frac{161}{36}\right)^2 \simeq \underline{1.97}$

c) La $Z = 7 - X$, da er $p(Y=i) = p(Z=i)$, dvs Y og Z har samme fordeling.

$E(Y) = E(Z) = E(7-X) = 7 - E(X) \simeq 7 - 4.47 = \underline{2.53}$

d) $Var(Y) = Var(Z) = Var(7-X) = (-1)^2\cdot Var(X) = Var(X) \simeq \underline{1.97}$

e) Det er kun den marginale fordelingen til $Z = 7 - X$ som er den samme som fordelingen til Y, og vi må her bruke simultan fordeling:

$E(XY) = 1\cdot 1\cdot 1/36 + 2\cdot 1\cdot 2/36 + 2\cdot 2\cdot 1/36 + (3\cdot 1 + 3\cdot 2)\cdot 2/36$

$+ (3\cdot 3 + 4\cdot 1 + 5\cdot 5 + 6\cdot 6)\cdot\frac{1}{36} + (4\cdot 1 + 4\cdot 2 + 4\cdot 3 + 5\cdot 1 + 5\cdot 2 + 5\cdot 3 + 5\cdot 4 + 6\cdot 1 + 6\cdot 2 + 6\cdot 3 + 6\cdot 4 + 6\cdot 5)\frac{2}{36}$

$= \frac{441}{36} = 12.25$

f) $E(X) \cdot E(Y) \simeq 4.47 \cdot 2.53 \simeq 11.30$

g) $P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)}$

$= \frac{2/36}{11/36} = 2/11 \simeq 0.18$

h) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$\simeq 12.25 - 11.30 \simeq 0.95$

3. Summen av alle sannsynligheter = 1

$0.1 + a + 0.2 + b + 0.1 + 0.3 = 1$

$a + b = 1 - 0.7 = 0.3$

$b = 0.3 - a$

Marginale:

$P_X(1) = 0.3 + a$

$P_X(2) = 0.4 + b = 0.7 - a$

$P_Y(1) = 0.1 + b = 0.4 - a$

$P_Y(2) = 0.1 + a$

$P_Y(3) = 0.5$

a) $E(X) = 1(0.3+a) + 2(0.7-a) = 1.7 - a$

b) $E(X^2) = 1^2(0.3+a) + 2^2(0.7-a) = 3.1 - 3a$

$\Rightarrow Var(X) = 3.1 - 3a - (1.7-a)^2 = -a^2 + 0.4a + 0.21$

c) $E(Y) = 1\cdot(0.4-a) + 2\cdot(0.1+a) + 3\cdot 0.5$

$= \underline{\underline{2.1+a}}$

d) $E(Y^2) = 1^2\cdot(0.4-a) + 2^2\cdot(0.1+a) + 3^2\cdot 0.5$

$= 5.3 + 3a$

$\Rightarrow Var(Y) = (5.3+3a) - (2.1+a)^2$

$= \underline{\underline{-a^2 - 1.2a + 0.89}}$

e) $E(XY) = 1\cdot 1\cdot 0.1 + 1\cdot 2\cdot a + 1\cdot 3\cdot 0.2$
$+ 2\cdot 1\cdot(0.3-a) + 2\cdot 2\cdot 0.1 + 2\cdot 3\cdot 0.3$

$= \underline{\underline{3.5}}$

f) $E(X)\cdot E(Y) = (1.7-a)(2.1+a) = \underline{\underline{3.57 - 0.4a - a^2}}$

g) $P(Y=2 \mid X=1) = \dfrac{P(Y=2, X=1)}{P(X=1)} = \underline{\underline{\dfrac{a}{0.3+a}}}$

h) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$= 3.5 - (3.57 - 0.4a - a^2) = \underline{\underline{a^2 + 0.4a - 0.07}}$

4. $Cov(X,Y) = a^2 + 0.4a - 0.07 = 0$

$$a = \frac{-0.4 \pm \sqrt{0.4^2 - 4(-0.07)}}{2}$$

$$= -0.2 \pm \sqrt{0.11}$$

$0 \le a \le 0.3$: $a = \sqrt{0.11} - 0.2 \simeq 0.13$ gir $Cov(X,Y) = 0$.

X, Y uavhengige $\Rightarrow p(1,3) = p_X(1) \cdot p_Y(3)$

$\Updownarrow$

$Cov(X,Y) = 0$

$\Downarrow$

$a \simeq 0.13$

$0.2 = (0.3+a) \cdot 0.5$

$0.2 = 0.15 + 0.5a$

Dette er ikke tilfellet om $a \simeq 0.13$, derfor kan ikke X og Y være uavhengige

5.

a) $\mu(R_1) > \mu(R_2)$, $\sigma(R_1) < \sigma(R_2)$ } R_1 har høyere forventet avkastning og lavere varians

$\Downarrow$

Velger R_1

b) $R = 0.5R_1 + 0.5R_2$ avkastning til porteføljen

$E(R) = 0.5 \cdot 1 + 0.5 \cdot 0.8 = 0.9$

$Var(R) = 0.5^2 \cdot Var(R_1) + 0.5^2 \cdot Var(R_2) + 2 \cdot 0.5 \cdot 0.5 \cdot Cov(R_1, R_2)$

$= 0.25 \cdot 0.1^2 + 0.25 \cdot 0.12^2 + 0.5 \cdot 0.1 \cdot 0.12 \cdot (-0.80)$

$\simeq 0.0013 \Rightarrow \sigma_R \simeq \sqrt{0.0013} \simeq 0.036$

c) $R = 0.8R_1 + 0.2R_2$

$E(R) = 0.8 \cdot 1 + 0.2 \cdot 0.8 = 0.96$

$Var(R) = 0.8^2 \cdot 0.1^2 + 0.2^2 \cdot 0.12^2 + 2 \cdot 0.8 \cdot 0.2 \cdot 0.1 \cdot 0.12 (-0.8)$

$\simeq 0.0039 \Rightarrow \sigma_R \simeq \sqrt{0.0039} \simeq 0.062$

d) $R = aR_1 + (1-a)R_2 \quad , \quad 0 \leq a \leq 1$

$E(R) = a \cdot 1 + (1-a) \cdot 0.8 = 0.8 + 0.2a$

$Var(R) = a^2 \cdot 0.1^2 + (1-a)^2 \cdot 0.12^2 + 2a(1-a) 0.1 \cdot 0.12 (-0.8)$

$= 0.01 a^2 + 0.0144 (1 - 2a + a^2) - 0.0192 (a - a^2)$

$= 0.0436 a^2 - 0.0048 a + 0.0144$

$\Rightarrow \sigma_R = \sqrt{0.0436 a^2 - 0.0048 a + 0.0144}$

6. $E(aX + bY) = a E(X) + b E(Y)$

$Var(aX + bY) = a^2 \cdot Var(X) + b^2 Var(Y) + 2ab \cdot Cov(X, Y)$

X, Y uncorr. $\Rightarrow Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$

since $Cov(X, Y) = 0$.

7. a) $X = X_1 + X_2 + X_3 + X_4$

$E(X) = \mu \cdot 4 = \underline{\underline{4\mu}}$

$Var(X) = \sigma^2 \cdot 4 = 4\sigma^2 \Rightarrow \sigma_X = \underline{\underline{2\sigma}}$

b) $X = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$

$E(X) = \frac{1}{4}\mu \cdot 4 = \underline{\underline{\mu}}$

$Var(X) = \left(\frac{1}{4}\right)^2 \cdot \sigma^2 \cdot 4 = \sigma^2/4 \Rightarrow \sigma_X = \underline{\underline{\sigma/2}}$

c) $X = \frac{1}{8}X_1 + \frac{1}{4}X_2 + \frac{1}{8}X_3 + \frac{1}{2}X_4$

$E(X) = \frac{1}{8}\mu + \frac{1}{4}\mu + \frac{1}{8}\mu + \frac{1}{2}\mu = \underline{\underline{\mu}}$

$Var(X) = \left(\frac{1}{8}\right)^2\sigma^2 + \left(\frac{1}{4}\right)^2\sigma^2 + \left(\frac{1}{8}\right)^2\sigma^2 + \left(\frac{1}{2}\right)^2\sigma^2$

$= \left(\frac{1}{64} + \frac{1}{16} + \frac{1}{64} + \frac{1}{4}\right)\sigma^2$

$= \frac{2+4+16}{64}\sigma^2 = \frac{22}{64}\sigma^2 = \frac{11}{32}\sigma^2$

$\Rightarrow \sigma_X = \underline{\underline{\sqrt{\frac{11}{32}} \cdot \sigma}}$